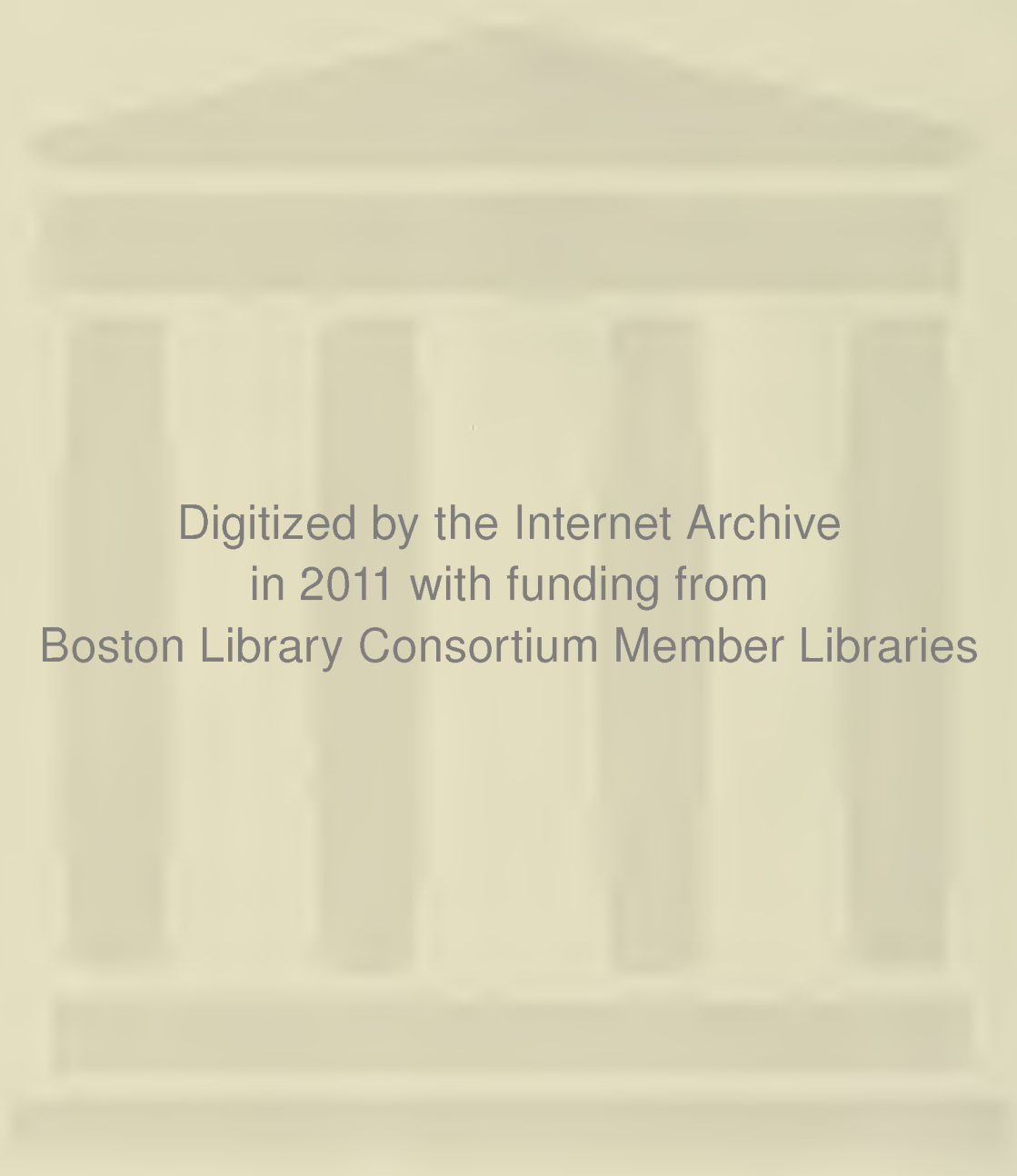


MIT LIBRARIES



3 9080 00856 7403





Digitized by the Internet Archive
in 2011 with funding from
Boston Library Consortium Member Libraries

<http://www.archive.org/details/necessarysuffici00smit2>

HB31
.M415

50

Dewey

no.

94-17

**working paper
department
of economics**

NECESSARY AND SUFFICIENT CONDITIONS
FOR THE PERFECT FINITE HORIZON FOLK THEOREM

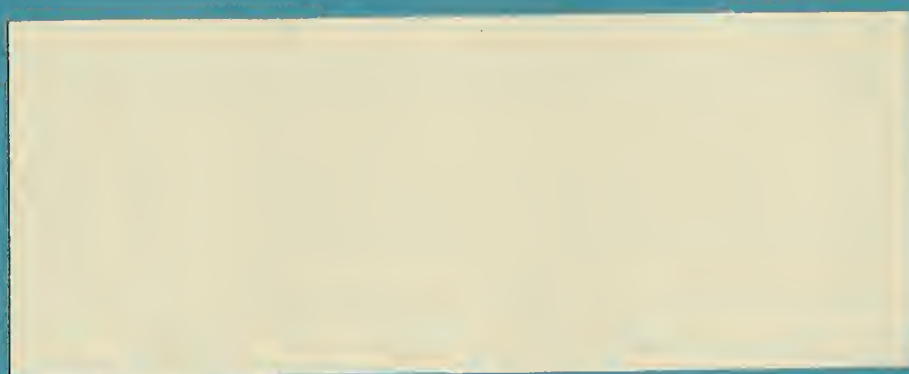
LONES SMITH

94-17 (93-6 Rev.)

May 1994

**massachusetts
institute of
technology**

**50 memorial drive
cambridge, mass. 02139**



NECESSARY AND SUFFICIENT CONDITIONS
FOR THE PERFECT FINITE HORIZON FOLK THEOREM

LONES SMITH

94-17 (93-6 Rev.)

May 1994

M.I.T. LIBRARIES

JUL 12 1994

RECEIVED

Forthcoming in *Econometrica*

Necessary and Sufficient Conditions
for the Perfect Finite Horizon Folk Theorem

by Lones Smith*

Department of Economics
Massachusetts Institute of Technology
Cambridge, MA 02139

May 23, 1994

1. INTRODUCTION

For a given n -player normal form game G , let $G(\delta, T)$ be the T -fold repeated game where the objective function is the average discounted sum of stage payoffs. For this finitely-repeated game, the *perfect folk theorem* is said to hold if the set of subgame perfect equilibrium (SPE) payoffs includes any feasible and strictly individually rational payoff vector of G for large enough $T < \infty$ and $\delta < 1$. Benoit and Krishna (1985) [hereafter, BK] produced a perfect folk theorem for $G(\delta, T)$ that obtains when (i) G satisfies the sufficient conditions for the infinite-horizon folk theorem — namely, the full-dimensionality condition of Fudenberg and Maskin (1986), and (ii) each player has distinct Nash payoffs in G .

Recently, Abreu, Dutta and Smith (1993) discovered an (essentially) iff condition for (i) — *non-equivalent utilities* (NEU) — that neatly supplants full-dimensionality. Wen (1993) subsequently examined games possibly violating NEU, and proved a revised general ‘folk theorem’. Here I provide a simple, easily verifiable, and *necessary and sufficient* condition for the finite-horizon folk theorem that replaces (ii): The intuitive condition, *recursively distinct Nash payoffs*, only insists that players’ behavior be *iteratively* leveraged near the end of the repeated game.

The primary contribution of this note is purely conceptual: I wish to finish the work of BK,¹ and thus complete the perfect information folk theorem program. Indeed, just as NEU only differed from full-dimensionality by a nongeneric class of games, so too, within the class of stage games with *recursively* distinct Nash payoffs, the measure without distinct Nash payoffs for all players is zero. But conceptual clarity is not without its own reward, as I later provide a necessary and sufficient condition for the finite-horizon *Nash* folk theorem. I also recast BK in Wen’s more encompassing framework.² In so doing, I happen upon a simple new proof of the necessity of NEU in Abreu et al. (1993), and of what ‘necessity’ means in Wen’s tiered ‘folk’ theorems more generally.

The logic of the BK folk theorem, as best exemplified in Krishna (1989) or Smith (1990/92) is rather simple: Late in the repeated game, because all players have distinct Nash payoffs, the behavior of any one of them can be *leveraged* by threatening to finish off with a fixed number (say S) of plays of that player’s worst Nash payoff, rather than cycle through all his best Nash payoffs. Away from the end of the game, the infinite-horizon punishments work perfectly well. And because S is fixed independent of the horizon length, the effect on the average payoff can be made arbitrarily small. This essentially is their proof.

Rather than formally define my condition, I first motivate it with the following three-player example game $\tilde{G}$.

¹It should be noted that BK conjecture under weak conditions that only one player need have distinct payoffs. This paper might in part be seen as proving a precise and rigorous statement of this conjecture.

²I am grateful to a referee for pressing me on this point.

2,2,3	2,2,2	2,2,2
2,2,2	2,2,2	2,2,2
2,2,2	2,2,2	-1,-1,0

2,1,-1	0,-1,-1	0,-1,-1
-1,0,-1	-1,-1,-1	-1,-1,-1
-1,0,-1	-1,-1,-1	-1,-1,-1

In this game, 1 chooses rows (actions U, M, D), 2 chooses columns (actions ℓ, m, r), and 3 chooses matrices (actions L, R). Player 3 strictly prefers L to R , while action D (resp. r) is weakly dominated for player 1 (resp. player 2). Thus it is easy to see that the only Nash (pure or mixed) payoffs are all convex combinations of $(2, 2, 2)$ and $(2, 2, 3)$. Furthermore, the minimax payoff is 0 for all three players.³

Because players 1 and 2 have unique Nash payoffs, $\tilde{G}$ does not satisfy the BK condition (ii). Nonetheless, a folk theorem does obtain! For $\tilde{G}$ satisfies NEU, and thus the standard infinite-horizon folk theorem applies to $\tilde{G}$. Next, player 3 enjoys the distinct (extremal) Nash payoffs of 2 and 3, so that his behavior is leveraged near the end of the game: I need only threaten to switch from (U, ℓ, L) to (M, ℓ, L) for the S -period phase. By choosing S large enough, 3 is willing to play R for as many periods, say S' , as I wish just prior to this Nash phase. When player 3 plays R irrespective of what players 1 and 2 do, a new game $\tilde{G}(R)$ is *induced* for players 1 and 2. It has the unique Nash equilibrium payoff vector $(2, 1)$. So player 2's Nash payoff from $\tilde{G}(R)$ is 1, which differs from his unique Nash payoff in $\tilde{G}(R)$ (i.e. when player 3 plays L). I have now leveraged the behavior of player 2 near the end of the game! Iterate this process. By choosing S' (and by implication S) large enough, I can induce player 2 to play ℓ, m , or r for as many periods, say S'' , as I wish just prior to this penultimate 'Nash' phase. In particular, players 2 and 3 are willing to play (r, R) irrespective of what player 1 does, yielding a new (one-player) game $\tilde{G}(r, R)$ for player 1. It has the unique optimal payoff of 0, which differs from 1's unique optimal payoff of 2 in $\tilde{G}(\ell, L)$. I have now also leveraged the behavior of player 1 near the end of the game! That $\tilde{G}$ satisfies a folk theorem now follows by the same proof as before.

I summarize the above procedure by saying that $\tilde{G}$ has *recursively distinct Nash payoffs*. If a game G satisfies NEU, it is *sufficient* for the BK result: So long as such a recursive procedure eventually leverages the behavior of all players, then a perfect folk theorem obtains. And my new condition is *necessary* too for the general perfect finite-horizon 'folk theorem'. Namely, if for any such chain of recursive reductions, I cannot leverage the behavior of all players, then a 'folk theorem' does not obtain.

The intuition behind the necessity is also rather simple. For any given horizon length T , a player's set of SPE payoffs is either point-valued, or it is not. Evidently, if the behavior of a player can be leveraged as above, then his SPE payoffs are multivalued for large enough T , while if his behavior cannot be leveraged, then he entertains a unique equilibrium

³But observe that no one player can simultaneously minimax the other two, a fact of some importance later on in my remarks on the necessity of NEU.

payoff for all T . Clearly, in the latter case, I cannot possibly hope for anything more. For instance, in $\tilde{G}$ with no discounting, player 3 has a multivalued SPE payoff set for all T , while players 2 and 1 only receive distinct SPE payoffs for $T \geq 5$ and $T \geq 19$, respectively.

The special payoffs in $\tilde{G}$ speak to the genericity of the distinct Nash payoff requirement when the folk theorem obtains. Indeed, if one player has distinct Nash payoffs, then the game has at least two pure and generically a third *mixed* Nash equilibrium. Thus, for generic games satisfying my condition, all players enjoy distinct Nash payoffs.

Section 2 focuses on the stage game, and describes my iterative procedure. I dwell on sufficiency (the folk theorem) in section 3.1, and necessity (in all respects) in section 3.2.

2. THE STAGE GAME

2.1 Basic Definitions

Let $G = \langle A_i, \pi_i; i = 1, \dots, n \rangle$ be a finite normal form n -player game, where A_i is player i 's finite set of actions, and $A = \times_{i=1}^n A_i$. Let player i 's utility function be $\pi_i : A \rightarrow \mathbb{R}$, and set $\pi(a) \equiv (\pi_1(a), \dots, \pi_n(a))$. Let M_i be player i 's mixed strategy set, with $M = \times_{i=1}^n M_i$. Simply write $\pi_i(\mu)$ for i 's expected payoff under the mixed strategy $\mu = (\mu_1, \dots, \mu_n) \in M$.

The game G has *non-equivalent utilities* (NEU) if no two players' von Neumann Morgenstern utility functions are equivalent, i.e. for all i and j , $\pi_i(\cdot)$ is not a positive affine transformation of $\pi_j(\cdot)$. Denote the set of players as $\mathcal{I} = \{1, 2, \dots, n\}$. Following Wen (1993), let $\{\mathcal{I}_b \subseteq \mathcal{I}, b = 1, \dots, B\}$ be the coarsest partition of $\mathcal{I}$ such that any two players in the same $\mathcal{I}_b$ have equivalent utilities; denote by $\mathcal{I}(i)$ all players with equivalent utilities to i . Normalize payoffs so that $\pi_i(\cdot) \equiv \pi_j(\cdot)$ for all $j \in \mathcal{I}(i)$. The *effective minimax* payoff level $\pi_i(w^i) = \min_a \max_{j \in \mathcal{I}(i)} \max_{a_j} \pi_i(a_j, a_{-j})$ of player i is the best payoff that anyone in $\mathcal{I}(i)$ can guarantee himself, i.e. the greatest minimax payoff over all players in $\mathcal{I}(i)$.⁴

Normalize $\pi_i(w^i) = 0$ for all i . Define $F = \text{co}\{\pi(\mu) : \mu \in M\}$, and call $F^* = \{w \in F : w_i > 0, \text{ for all } i\}$ the *feasible and (strictly) rational* payoff set. To sidestep trivialities, let $F^* \neq \emptyset$. Note that under NEU, w^i reduces to the standard minimax strategy for player i , and F^* the feasible and (strictly) *individually rational* payoff set.

⁴Wen (1993) does not describe it as such, but it is true. Indeed, w^i is certainly a minimax profile for some player $j \in \mathcal{I}(i)$. To see that $\pi_i(w^i)$ weakly exceeds any minimax payoff for all $j \in \mathcal{I}(i)$, let $\hat{\pi}_j(a) \equiv \max_{a'_j} \pi_j(a'_j, a_{-j})$ for $j \in \mathcal{I}$. Analogous to the fact that the minimax $\geq$ maximin in constant sum games,

$$\pi_i(w^i) = \min_a \max_{j \in \mathcal{I}} \hat{\pi}_j(a) \geq \max_{j \in \mathcal{I}} \min_a \hat{\pi}_j(a) = \max_{j \in \mathcal{I}} \left(\min_{a_{-j}} \max_{a_j} \pi_j(a_j, a_{-j}) \right).$$

2.2 Recursively Distinct Nash Payoffs

Given a subset of players⁵ $\mathcal{J}' = \{j_1, j_2, \dots, j_m\} \subset \mathcal{I}$ and their (possibly mixed) actions

$$a_{\mathcal{J}'} \equiv (a_{j_1}, a_{j_2}, \dots, a_{j_m}) \in M_{j_1} \times M_{j_2} \times \dots \times M_{j_m} \equiv M_{\mathcal{J}'}, \quad (1)$$

let $G(a_{\mathcal{J}'})$ be the induced $(n - m)$ -player game for players $\mathcal{I} \setminus \mathcal{J}'$ obtained from G when the actions of players $\mathcal{J}'$ are fixed to $a_{\mathcal{J}'}$.

Define a *Nash decomposition* of G as an increasing sequence of $h \geq 1$ nonempty subsets of players from $\mathcal{I}$, namely

$$\{\emptyset = \mathcal{J}_0 \subset \mathcal{J}_1 \subset \mathcal{J}_2 \subset \dots \subset \mathcal{J}_h \subseteq \mathcal{I}\}, \quad (2)$$

so that for $g = 1, \dots, h$, actions $e_{\mathcal{J}_{g-1}}, f_{\mathcal{J}_{g-1}} \in M_{\mathcal{J}_{g-1}}$ exist with a pair of Nash payoff vectors $y(e_{\mathcal{J}_{g-1}})$ of $G(e_{\mathcal{J}_{g-1}})$ and $y(f_{\mathcal{J}_{g-1}})$ of $G(f_{\mathcal{J}_{g-1}})$ different *exactly* for players in $\mathcal{J}_g \setminus \mathcal{J}_{g-1}$, i.e.

$$y(e_{\mathcal{J}_{g-1}})_i \neq y(f_{\mathcal{J}_{g-1}})_i \quad (3)$$

for all $i \in \mathcal{J}_g \setminus \mathcal{J}_{g-1}$. For instance, players in $\mathcal{J}_1$ have distinct Nash payoffs in G .

The game G has *recursively distinct Nash payoffs* if there is a Nash decomposition with $\mathcal{J}_h = \mathcal{I}$. So if, as in BK, all players have distinct Nash payoffs in G , then this condition holds. The converse is not true, as illustrated earlier. I later consider games G that do *not* have recursively distinct Nash payoffs, i.e. $\mathcal{J}_h = \mathcal{I}$ is impossible. An ambiguity may then arise, since Nash decompositions need not be unique. It is not inconceivable that the union of *all* Nash decompositions could include every player. In fact, this cannot occur.⁶

Lemma *There is a well-defined maximal set of players $\mathcal{J}^* \subseteq \mathcal{I}$ who have recursively distinct Nash payoffs.*

3. FINITELY-REPEATED GAMES

3.1 Sufficiency of Recursively Distinct Nash Payoffs

I shall analyze finitely-repeated games with *perfect monitoring*, allowing each player to condition his current actions on the past actions of all players.

Let $\alpha_i = (\alpha_{i1}, \dots, \alpha_{iT})$ be a behavior strategy for player i , and $\pi_{it}(\alpha)$ his expected payoff in period t with the strategy profile α . Player i 's objective function in $G(\delta, T)$ is the expected discounted sum of his payoffs: $\frac{1-\delta}{1-\delta^T} \sum_1^T \delta^{t-1} \pi_{it}(\alpha)$. The set of SPE payoffs is $V(\delta, T)$.

⁵Throughout, $A \subset B$ means that A is a *strict* subset of B , i.e. $A \subseteq B$ and $A \neq B$.

⁶The proof of this result is omitted; see my (1994) working paper.

Beyond replacing the distinct Nash payoff requirement, Theorem 1 differs from BK's Theorem 3.7 in three ways. First, it admits payoff discounting, where δ and T can vary independently over the relevant range. Second, it is a more general 'folk theorem', as pursued by Wen (1993), which will imply BK's result with NEU. Third, unlike the (more intuitive) use of long deterministic cycles in BK, *public randomization* is used: In every period, players can condition on the outcome of a publicly observed exogenous continuous random variable.⁷ Just as in BK, I shall assume that players can observe deviations within the support of strictly mixed strategies. Alternatively, just assume that the folk theorem refers to the pure strategy minimax payoff level.⁸

Theorem 1 (The 'Folk Theorem') *Suppose that the stage game G has recursively distinct Nash payoffs. Then for the finitely-repeated game $G(\delta, T)$, $\forall u \in F^*$ and $\forall \varepsilon > 0$, $\exists T_0 < \infty$ and $\delta_0 < 1$ so that $T \geq T_0$ and $\delta \in [\delta_0, 1] \implies \exists v \in V(\delta, T)$ with $\|v - u\| < \varepsilon$.*

Proof: Fix a Nash decomposition (2) for which inequality (3) obtains, and define

$$c_g = \min_{i \in \mathcal{J}_g \setminus \mathcal{J}_{g-1}} \|y(e_{\mathcal{J}_{g-1}})_i - y(f_{\mathcal{J}_{g-1}})_i\| > 0$$

for $g = 1, 2, \dots, h$. Let the $\rho > 0$ be the largest payoff range (i.e. the difference between best and worst payoffs) for any player in G , and $\psi_g(k)$ the least even number above $2k\rho/c_g$. Saving on notation, let y^g denote $y(e_{\mathcal{J}_{g-1}})$ in even periods and $y(f_{\mathcal{J}_{g-1}})$ in odd ones. Further, let $z^{g,i}$ be the *less preferred* Nash payoff vector amongst $y(e_{\mathcal{J}_{g-1}})$ and $y(f_{\mathcal{J}_{g-1}})$ for player $i \in \mathcal{J}_g \setminus \mathcal{J}_{g-1}$. Since for all $i \in \mathcal{J}_g \setminus \mathcal{J}_{g-1}$,

$$-k\rho + \psi_g(k)y_i^g > \psi_g(k)z_i^{g,i}, \quad (4)$$

it follows that without payoff discounting, any player i has a strict incentive to conform to k consecutive periods of any action profile followed by $\psi_g(k)$ periods of y^g , if deviations are punished by switching each y^g to $z^{g,i}$.

For any $m > 0$, recursively define $s_h(m) = \psi_h(m)$ and

$$s_g(m) = \psi_g(m + s_{g+1}(m) + \dots + s_h(m))$$

for $g = h-1, h-2, \dots, 1$. Define $t_0(m) = 0$, and $t_g(m) = s_1(m) + \dots + s_g(m)$ for

⁷One may also take advantage of the correlating device to produce a nearly *exact*, rather than an *approximate*, folk theorem. That is, one can use the correlation device to do away with the ε -approximation for all payoff vectors not on the boundary of F . I omit such a laborious exercise.

⁸This assumption is not necessary to detect deviations from the minimax phase (step 3, below), as noted in my (1994) working paper. But for the two recursively Nash phases, no such workaround exists. It would thus be interesting to know whether Theorem 2 goes through when $\mathcal{J}^*$ is defined given pure actions (1), in which case this would not be a concern.

$g = 1, \dots, h$. The T -period equilibrium outcome sequence is

$$a, \dots, a; y^h, \dots, y^h; \dots; y^1, \dots, y^1$$

where a is played for $T - t_h(q + r)$ periods,⁹ and y^g is played for $s_g(q + r)$ periods. Fix $\varepsilon > 0$. If v is the δ -discounted average payoff vector, then $\|v - u\| < \varepsilon$ for big enough δ and T .

Abreu et al. (1993) have established the existence of feasible payoff vectors $x^1, \dots, x^n$ such that for all $i \neq j$, $x^i \gg 0$ (strict IR), $x_i^i < x_i^j$ for all $j \notin \mathcal{I}(i)$ (payoff asymmetry), and $x_i^i < u_i$ (target payoff domination).

I now explicitly describe the players' strategies which support this equilibrium.¹⁰ For ease of exposition, *late* deviations are those occurring during the final $q + r + t_h(q + r)$ periods of the repeated game; all others are called *early* deviations.

1. MAIN PATH. Play a until period $T - t_h(q + r)$. [If any i deviates early, start 3; if some player in $\mathcal{J}_{g'}$ deviates late, start 5.]
2. GOOD RECURSIVE NASH PHASE. For $g = h, \dots, 1$: Play y^g in periods $T - t_g(q + r) + 1, \dots, T - t_{g-1}(q + r)$. [If some $i \in \mathcal{J}_{g'}$ deviates late, where $g' < g$, start 5.]
3. MINIMAX PHASE. Play w^i for q periods. [If $j \notin \mathcal{I}(i)$ deviates, start 4.] Set $j \leftarrow i$.
4. REWARD PHASE. Play x^j for r periods. [If any i deviates early, restart 3; if some $i \in \mathcal{J}_{g'}$ deviates late, start 5.] Then return to step 1 or 2.
5. BAD RECURSIVE NASH PHASE. Play $z^{g',i}$ until period $T - t_{g'-1}(q + r)$. [If $j \in \mathcal{J}_{g''}$ deviates, where $g'' < g'$, set $g' \leftarrow g''$ and $i \leftarrow j$ and restart 5.] Then go to step 2.

It is straightforward (or see my (1994) working paper) to verify that these strategies constitute an SPE for some q, r and big enough δ_0 and T_0 .¹¹ $\diamond$

3.2 Necessity of Recursively Distinct Nash Payoffs

If a game G does not have recursively distinct Nash payoffs, the consequences are rather stark. The following result has a flavour of the “zero-one” laws of probability

⁹Note that q and r are implicitly defined in steps 3 and 4 below.

¹⁰Below, i and j denote arbitrary players, and g, g' , and g'' arbitrary indices in $\{1, 2, \dots, h\}$. For clarity, I shall use the simple notation $j \leftarrow i$ to mean “assign j the value i .” Also, steps always follow sequentially, unless otherwise indicated. Bracketed remarks refer to off-path play, i.e. following deviations.

¹¹Absent public randomization, the simple direct proof presented above would require two modifications: First, the target outcome a would have to be replaced by an approximating finite outcome profile. Second, ditto for each x^i vector. Finally, to my knowledge, public randomization is essential if one wishes to work with the mixed minimax strategy, which I noted earlier was possible. See for instance, Abreu et al. (1993).

theory. Namely, as $T \rightarrow \infty$, either $V(\delta, T)$ tends to the strictly individually rational payoff set F^* , or some players receive a *unique* SPE payoff. There is no middle ground.

Theorem 2 *For any $T < \infty$ and any $\delta \in (0, 1]$, players in $\mathcal{I} \setminus \mathcal{J}^*$ receive a payoff in any SPE of $G(\delta, T)$, equal to their unique Nash equilibrium payoff of G .¹²*

Proof: Every player in $\mathcal{I} \setminus \mathcal{J}^*$ has a unique Nash and hence SPE payoff in $G(\delta, 1) \equiv G$. Assume that everyone in $\mathcal{I} \setminus \mathcal{J}^*$ has a unique SPE payoff in $G(\delta, T)$ equal to his Nash payoff in G , for $T = 1, \dots, T_0$. Then in the first period of $G(\delta, T_0 + 1)$, players in $\mathcal{I} \setminus \mathcal{J}^*$ must play a Nash equilibrium of $G(e_{\mathcal{J}^*})$, for any $e_{\mathcal{J}^*} \in M_{\mathcal{J}^*}$, because they have a unique SPE continuation payoff by assumption. By induction, the result obtains for all T . $\diamond$

Remarks:

1. This stark necessity result contrasts markedly with the partial failure of NEU, where we need only replace the minimax payoff level with the effective minimax payoff level.

2. Let $\tilde{F}^* = \{v \in F^* \mid v_i > \min_a \max_{j \in \tilde{\mathcal{I}}(i)} \max_{a_j} \pi_i(a_j, a_{-j}) \ \forall i\}$, where the partition $\{\tilde{\mathcal{I}}_b\}$ strictly refines $\{\mathcal{I}_b\}$.¹³ If $(\star)$ for no such partition $\{\tilde{\mathcal{I}}_b\}$ does there exist any strategy profile separately holding two or more players in any $\mathcal{I}_b$ to their worst feasible payoff in $\tilde{F}^*$, then necessarily $\pi_i(w^i) > \min_a \max_{j \in \tilde{\mathcal{I}}(i)} \max_{a_j} \pi_i(a_j, a_{-j})$, for $\tilde{\mathcal{I}}(i) \subset \mathcal{I}(i)$. But Wen (1993) proves that all SPE payoffs Pareto dominate the effective minimax outcome.¹⁴ This yields ‘necessity’ in another respect: The folk theorem fails when F^* is replaced by $\tilde{F}^*$. By corollary, *the necessity result of Abreu et al. (1993) follows!* For under NSM (a particularization of $(\star)$), if NEU fails, then $\{\mathcal{I}_b\}$ admits a strict refinement, and the (standard) finite- or infinite-horizon folk theorem cannot obtain!

3. Recursively distinct Nash payoffs is also the necessary and sufficient condition for the finite horizon Nash folk theorem, due to Benoit and Krishna (1987).¹⁵

¹²Note that this theorem says nothing about the payoff set of players in $\mathcal{J}^*$.

¹³That is, $\emptyset \neq \tilde{\mathcal{I}}(i) \subseteq \mathcal{I}(i)$ for all i , and $\tilde{\mathcal{I}}(i) \subset \mathcal{I}(i)$ for some i .

¹⁴Unlike his folk theorem, this result of his actually doesn’t rely on observable mixed strategies. And it holds for finitely-repeated games too.

¹⁵I am grateful to a referee for suggesting this. For a proof of sufficiency, consider the following strategies: Let the equilibrium path follow steps 1 and 2. Off-path, retain the minimax phase, step 3, but allow it to last until the end of the game, and omit any reference to play after deviations, including step 4 (as it is not needed in a Nash equilibrium); finally, retain step 5, but omit the references to off-path play. To see necessity, just observe that Theorem 2 still obtains if SPE is replaced by Nash equilibrium.

References

- Abreu, Dilip, and Prajit Dutta (1991) 'Folk Theorems for Discounted Repeated Games: a New Condition.' University of Rochester mimeo
- Abreu, Dilip, Prajit Dutta, and Lones Smith (1993) 'The Folk Theorem for Repeated Games: a NEU Condition.' Forthcoming in *Econometrica*
- Benoit, Jean-Pierre, and Vijay Krishna (1985) 'Finitely Repeated Games.' *Econometrica* 53, 905–922
- (1987) 'Nash Equilibria of Finitely Repeated Games.' *International Journal of Game Theory* 16, 197–204
- Fudenberg, Drew, and Eric Maskin (1986) 'The Folk Theorem in Repeated Games with Discounting or Incomplete Information.' *Econometrica* 54, 533–554
- Krishna, Vijay (1989) 'The Folk Theorems for Repeated Games.' mimeo 89-003, Harvard Business School
- Smith, Lones (1990/92) 'Folk Theorems: Two-Dimensionality is (Almost) Enough.' MIT mimeo
- Wen, Quan (1993) 'The 'Folk Theorem' for Repeated Games with Complete Information.' Forthcoming in *Econometrica*

4. APPENDIX: OMITTED MATERIAL

4.1 Proof of Lemma

If for some Nash decomposition I have $\mathcal{J}_h = \mathcal{I}$, then unambiguously $\mathcal{J}^* = \mathcal{I}$, and I am done.

Otherwise, let $\mathcal{E}_i(G)$ be the set of all Nash payoffs for player i in the game G . Consider the alternative *unique* nested sequence of h' nonempty sets $\{\tilde{\mathcal{J}}_g\}$ satisfying

$$\{\emptyset = \tilde{\mathcal{J}}_0 \subset \tilde{\mathcal{J}}_1 \subset \tilde{\mathcal{J}}_2 \subset \cdots \subset \tilde{\mathcal{J}}_{h'} \subseteq \mathcal{I}\},$$

given by the following (well-defined) recursion: Given $\tilde{\mathcal{J}}_g$, the set $\tilde{\mathcal{J}}_{g+1} \setminus \tilde{\mathcal{J}}_g$ is all players $i \in \mathcal{I} \setminus \tilde{\mathcal{J}}_g$ for whom $\{\mathcal{E}_i(G(a_{\tilde{\mathcal{J}}_g})), a_{\tilde{\mathcal{J}}_g} \in M_{\tilde{\mathcal{J}}_g}\}$ does not consist of a singleton. If no such players $\tilde{\mathcal{J}}_{g+1} \setminus \tilde{\mathcal{J}}_g$ exist, set $h' = g$.

Note that it may not be possible to find only two Nash equilibria providing distinct payoffs simultaneously for all players $i \in \tilde{\mathcal{J}}_{g+1} \setminus \tilde{\mathcal{J}}_g$. Still, I can show that $\tilde{\mathcal{J}}_{h'} = \mathcal{J}_h$ for all Nash decompositions (1). For it is clear that $\mathcal{J}_h \subseteq \tilde{\mathcal{J}}_{h'}$. Moreover, for any such nested sequence (1), all players in any $\tilde{\mathcal{J}}_{g'}$ are eventually included in *some* $\mathcal{J}_g$. For if $\{\mathcal{E}_i(G(a_{\tilde{\mathcal{J}}_{g'}})), a_{\tilde{\mathcal{J}}_{g'}} \in M_{\tilde{\mathcal{J}}_{g'}}\}$ does not consist of a singleton, then neither does $\{\mathcal{E}_i(G(a_{\tilde{\mathcal{J}}_{g''}})), a_{\tilde{\mathcal{J}}_{g''}} \in M_{\tilde{\mathcal{J}}_{g''}}\}$ if $\tilde{\mathcal{J}}' \subset \tilde{\mathcal{J}}''$.

4.2 Omitted Parts of Proof of Theorem 1

Step 1: Observable Mixed Strategies

I proceed recursively as I ensure subgame perfection. First note that *I only check that all deterrents are strict by some positive margin, say 1, without discounting*. Given continuity of discounted sums in δ , they will thus remain strict for any level of discounting $\delta \in [\delta_0, 1]$, for some $\delta_0 < 1$. I shall check that after no history consisting of isolated deviations does a one-shot deviation yield a higher payoff than the strategy profile; by the “unimprovability” criterion, that profile will be a subgame perfect equilibrium.

I now select q and r . In light of $x_i^i > 0$, let q satisfy¹⁶

$$w_i^i + qx_i^i > \beta_i^i + 1 \tag{4}$$

for all i , where β^i (resp. w^i) is the best (resp. worst) payoff vector for player i in G . Since $x_i^i < u_i$ too, (4) simultaneously renders steps 3 and 4 a strict deterrent to early deviations

¹⁶The inequality (4) — in particular, why the left-hand side is *not* $(q+1)x_i^i$ — reflects the fact that obeying the random correlating device generating x^i might sometimes require i to play his worst possible outcome.

from step 1, for any r . Next, step 4 deters deviations from step 3 if r is large enough that

$$qw_j^j + rx_j^i > \beta_j^j + rx_j^j + (q-1)u_j + 1$$

for all $j \neq i$. This is possible because $x_j^i > x_j^j$.

Next, I verify that step 5 deters all late deviations. Indeed, this is an immediate consequence of inequality (3), which upon substitution yields

$$\begin{aligned} & [q + r + s_{g+1}(q+r) + \dots + s_h(q+r)](-\rho) + s_g(q+r)y_i^g \\ &= \psi_g^{-1}(s_g(q+r))(-\rho) + s_g(q+r)y_i^g \\ &> s_g(q+r)z_i^{g,i} + 1 \end{aligned}$$

for all $i \in \mathcal{I}_g$, and $g = 1, 2, \dots, h$.

Given q and r as defined above, the above program is feasible if $T_0 \geq q + r + t_h(q+r)$. For $T \geq T_0$, each deterrent remains strict for all $\delta \in [\delta_0, 1]$, for some $\delta_0 < 1$. Finally, for $T_0 < \infty$ and $\delta_0 < 1$ large enough, the resulting average payoff vector $v \in F^*$ satisfies $\|u - v\| < \varepsilon$, since q and r are fixed.

Step 2: Unobservable Mixed Strategies

The argument is an adaptation of Abreu et al. (1993) — to which the reader is referred for a simple argument that I need only (below) consider deviations from m^i by those players $j \neq i$ for whom the projection of F onto the i - j coordinate plane has full dimension; for any other player k , play proceeds just as in step 1.¹⁷

There must exist payoff vectors $c^{ij} \in F^*$ such that: (a) i is indifferent between x^i and c^{ij} , or $x_i^i = c_i^{ij}$; (b) j is not indifferent between c^{ij} and x^i , or $c_j^{ij} \neq x_j^i$; and (c) j prefers c^{ij} to x^j , or $c_j^{ij} > x_j^j$. After step 4, payoff x^i in step 5 is probabilistically replaced by c^{ij} (via the public randomization device), where the probabilities are chosen to maintain the incentives of minimaxing players $j \neq i$, and player i is unaffected by the transitions given (a).

Let $p_t^{ij}(a_j)$ satisfy for all a_j, a'_j in the support of m_j^i , and $t \leq q$,

$$\begin{aligned} & \pi_j(a_j, m_{-j}^i) - \pi_j(a'_j, m_{-j}^i) = \\ & \delta^{q-t+1}(1 + \delta + \dots + \delta^{r-1})[p_t^{ij}(a'_j) - p_t^{ij}(a_j)](c_j^{ij} - x_j^i). \end{aligned} \quad (5)$$

That such *real* numbers $p_t^{ij}(\cdot)$ exist is a consequence of (b). That they can be chosen to be (non-unique) *probabilities* follows from this construction, for instance, when $c_j^{ij} > x_j^i$ in (b): Let $p_t^{ij}(a_j) = 0$, where action a_j yields the highest myopic payoff to player j (when the

¹⁷Below, if such k exist, the expression “ $j \neq i$ ” actually denotes $j \neq i, k$.

other players play m_{-j}^i). If action a'_j yields the lowest myopic payoff to player j , then after substituting $p_t^{ij}(a_j) = 0$ in (5), it will follow that $p_t^{ij}(a'_j) < 1/(n-1)q$ for large enough $\delta < 1$ and r . Furthermore, $p_t^{ij}(a''_j) < 1/(n-1)q$ for any larger δ and r , or any other action a''_j in the support of m_j^i . Also observe $c_j^{ij} > x_j^i$ implies that $p^{ij}(a''_j) \geq 0$ for all a''_j .

If the realized sequence of action profiles is $a^q = (a_1, a_2, \dots, a_q)$, then c^{ij} replaces x^i in step 4 with probability $p^{ij}(a^q) = \sum_1^q p_t^{ij}(a_{jt})$ for all $j \neq i$. By construction, $0 \leq p^{ij}(\cdot) < 1/(n-1)$ for each $j \neq i$, yielding a well-defined probabilistic transition function. As p_t^{ij} is *independent of actions by players $j \neq i$ and of j 's own actions in periods $s \neq t$* , (5) ensures that players are indifferent across the support of their minimaxing strategies. Finally, in light of (c), the punishments that previously supported step 5 still obtain.

MIT LIBRARIES



3 9080 00856740 3

Date Due

Lib-26-67

